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On the Miura map for discrete integrable systems

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Abstract. Gauss decomposition of the Lie algebra has been used to derive the Miura map for discrete completely integrable systems, which exactly reproduces the results obtained in a recent paper by Yang and Schmid. We have dealt with the cases of discrete mKdV, discrete sine–Gordon, discrete KdV and the Volterra system. Our approach has the advantage of yielding the Lax pair for the modified equation. It can also lead to a second stage of modified equation if the same procedure is repeated twice.

1. Introduction

Integrable nonlinear equations possess a very elegant property that sometimes a nonlinear transformation of the dependent variable may lead to another integrable system. The oldest example of such a transformation is the famous Miura map which connects the KdV equation to the mKdV equation. Such transformations have the potential of generating non-trivial solutions from trivial ones. Various methods have been suggested to derive such a mapping for different cases such as the KdV equation [1], the mKdV equation [2], the nonlinear Schrödinger equation [3], the Toda lattice [4], the discrete nonlinear Schrödinger equation [5] etc. Although a general framework has been set up for the continuous case, detailed analysis of the discrete equation is not available in the literature. Recently it has been shown by Yang and Schmid [6] that an extension of the methodology of Chen [7] can lead to a unified approach to the problem of Backlund and Miura maps for discrete integrable equation. On the other hand, a few years ago Dodd [8] introduced the idea of using Gauss decomposition of the gauge transformation of the Lax pair to derive the Miura map and Backlund transformation. He showed that such a procedure can lead to intermediate equations which are new integrable systems and, in the process, one discovers the Miura map. This procedure is, in fact, the matrix analogue of the factorization of the Schrödinger problem in the KdV case. Here in this communication we show that one can adopt the Dodd's procedure even in the case of discrete integrable systems, such as the discrete KdV, mKdV, sine–Gordon etc. It is shown that almost all the results obtained in [6] can be reproduced and, in addition, we are able to write down the Lax pair of the modified equation itself, which was not done in [6]. Furthermore this new Lax pair can be utilized again to generate second modified systems.

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2. Outline of the method

For a discrete integrable system the Lax pair can be written as

$$F_{n+1} = L_n(t,\lambda)F_n$$
$$\frac{\mathrm{d}F_n}{\mathrm{d}t} = V_n(t,\lambda)F_n. \tag{1}$$

The compatibility between these leads to

$$\frac{\mathrm{d}L_n}{\mathrm{d}t} + L_nV_n - V_{n+1}L_n = 0. \tag{2}$$

We assume that equation (1) admits a gauge transformation. Before adopting Dodd's procedure let us explain some notation that is essential for the ensuing calculation. Each of the matrices F_n, L_n etc are given two indices as $F_n^{i,j}$, $L_n^{i,j}$, where the first superscript labels a solution of the Lax equation and the second indicates the intermediate equation at a particular stage of Gauss factorization, so that the generalized form of the Lax equation can be written as

$$F_{n+1}^{i,j} = L_n^{i,j}F_n^{i,j} \tag{3}$$
$$F_{n,t}^{i,j} = V_n^{i,j}F_n^{i,j}. \tag{4}$$

In this notation $F_n^{0,0}$ denotes the solution of the starting Lax equation, $F_n^{0,1}$ denotes that of the first intermediate equation Lax equation and $F_n^{1,1}$ denotes another solution of the same intermediate equation connected to the previous one by an auto-Backlund transformation. In general $F_n^{i,j}$ is the ith solution of the jth intermediate Lax equation.

So we write the gauge transformation as

$$F_n^{i+1,j} = T_n^{i,j}F_n^{i,j}.$$

So from equations (3) and (4) we get

$$L_n^{i+1,j}T_n^{i,j} = T_{n+1}^{i,j}L_n^{i,j} \tag{5}$$

and

$$V_n^{i+1,j}T_n^{i,j} = T_{nt}^{i,j} + V_n^{i,j}T_n^{i,j}. \tag{6}$$

We now assume as in [8] that $T_n^{i,j}$ has a Gauss decomposition which can be written as

$$T_n^{i,j} = \Delta_n^{-i,j}\Delta_n^{+i,j} \tag{7}$$

where $\Delta_n^{+i,j}$ is upper unipotent and $\Delta_n^{-i,j}$ is lower triangular. So mimicking the continuous case we deduce that the equation to be satisfied by Δ_n^- and Δ_n^+ is

$$L_n^{i,j+1}\Delta_n^{+i,j} = \Delta_{n+1}^{+i,j}L_n^{i,j} \tag{8}$$

and if $\Delta_n^{+1,j}$ is non-singular we also get

$$\Delta_{n+1}^{-i,j}L_n^{i,j+1} = L_n^{i+1,j}\Delta_n^{-i,j} \tag{9}$$

along with

$$V_n^{i,j+1}\Delta_n^{+i,j} = \Delta_n^{+i,j}V_n^{i,j} + \Delta_{nt}^{+i,j}. \tag{10}$$

Let us now consider the case of a discrete mKdV equation for which

$$L_n^{0,0} = \begin{bmatrix} \lambda & q_n \\ -q_n & 1/\lambda \end{bmatrix} \tag{11}$$

$$V_n^{0,0} = \begin{bmatrix} \lambda^2 + q_{n-1}q_n & q_n\lambda + q_{n-1}/\lambda \\ -q_{n-1}\lambda - q_n/\lambda & 1/\lambda^2 + q_{n-1}q_n \end{bmatrix} \tag{12}$$

whereas the equation is written as

$$q_{nt} = (1+q_n^2)(q_{n+1} - q_{n-1}). \tag{13}$$

Let us assume

$$\Delta_n^{+0,0} = \begin{bmatrix} 1 & a_n \\ 0 & 1 \end{bmatrix} \qquad \text{and } L_n^{0,1} = \begin{bmatrix} b_n & c_n \\ d_n & f_n \end{bmatrix} \tag{14}$$

whence equation (8) yields

$$\begin{aligned} b_n &= \lambda - a_{n+1}a_n \\ b_n a_n + c_n &= q_n + \frac{a_{n+1}}{\lambda} \\ d_n &= -q_n \\ d_n a_n + f_n &= \frac{1}{\lambda}. \end{aligned} \tag{15}$$

From these equations if we demand that $c_n = 0$, we get

$$a_{n+1} = \frac{\lambda a_n - q_n}{1/\lambda + a_n q_n} \qquad \text{or } q_n = \frac{\lambda a_n - q_{n+1/\lambda}}{1 + a_{n+1}a_n} \tag{16}$$

whereas from equation (6) we deduce that a_n satisfies

$$a_{n,t} = \frac{a_{n+1} - a_n(\lambda^2 + 1/\lambda^2) - a_n^3}{1 + a_{n+1}a_n} + \frac{a_n^3 + a_n(\lambda^2 + 1/\lambda^2) - a_{n-1}}{1 + a_n a_{n-1}} \ldots. \tag{17}$$

It may be noted that equations (16) and (17) were deduced from the discrete symmetry considerations and Riccati-type equations in [6]. An interesting and important aspect of the present methodology is that in the course of our calculation we have deduced the Lax pair for the modified mKdV equation (17); these are given by $L_n^{0,1}$, and $V_n^{0,1}$ respectively. Written explicitly these are

$$L_n^{0,1} = \frac{1}{1 + a_n a_{n+1}} \begin{bmatrix} \lambda + \frac{a_{n+1}^2}{\lambda} & 0 \\ \frac{a_{n+1}}{\lambda} - \lambda a_n & \frac{1}{\lambda} + \lambda a_n^2 \end{bmatrix} \tag{18}$$

$$V_n^{0,1} = \begin{bmatrix} z_n & 0 \\ x_n & y_n \end{bmatrix} \tag{19}$$

where z_n, x_n, y_n are as follows

$$z_n = \frac{1}{1 + a_n a_{n-1}} \left[\lambda^2 + a_n^2 + \frac{\left(\lambda a_n - \frac{a_{n+1}}{\lambda}\right)\left(\lambda a_{n-1} - \frac{2a_n}{\lambda} - \frac{a_n^2 a_{n-1}}{\lambda}\right)}{(1 + a_n a_{n-1})} \right]$$

$$x_n = \frac{\left(\lambda^2 a_{n-1} + \lambda^2 a_n a_{n-1} a_{n+1} - a_n^2 a_{n+1} - \frac{a_{n+1}}{\lambda^2} + a_n^2 a_{n-1} - \frac{a_n a_{n+1} a_{n-1}}{\lambda^2}\right)}{(1 + a_n a_{n-1})(1 + a_n a_{n+1})}$$

$$\begin{aligned} y_n = \big\{&(1 + a_n a_{n-1} + a_n a_{n+1})\lambda^{-2} + a_n^3(a_{n-1} - a_{n+1}) + 2\lambda^2 a_{n-1} a_n (1 + a_n a_{n+1}) \\ &-(a_n^2 + a_{n-1}a_{n+1})\big\} / (1 + a_n a_{n-1})(1 + a_n a_{n+1}). \end{aligned}$$

Hence by employing the Gauss factorization technique it is possible to deduce not only the Miura map and the modified equation but also the Lax pair associated with the new equation.

3. Other discrete equations

Let us consider now a discrete sine–Gordon equation written as

$$q_{n+1,t} - q_{n,t} = 2h(\sin q_{n+1} + \sin q_n) \tag{20}$$

for which

$$L_n^{0,0} = \begin{bmatrix} \alpha_n & \beta_n \\ r_n & n \end{bmatrix}$$

where

$$\begin{aligned} \alpha_n &= \frac{\lambda}{2}(1+\cos q_n) + \frac{1}{2\lambda}(1-\cos q_n) \\ \beta_n &= \frac{\lambda}{2}\sin q_n - \frac{1}{2\lambda}\sin q_n \\ r_n &= \frac{\lambda}{2}\sin q_n - \frac{1}{2\lambda}\sin q_n \\ n &= \frac{\lambda}{2}(1-\cos q_n) + \frac{1}{2\lambda}(1+\cos q_n) \end{aligned} \tag{21}$$

along with

$$V_n^{0,0} = \begin{bmatrix} \frac{h(\lambda^2+1)}{\lambda^2-1} & -\omega_n \\ \omega_n & \frac{h(\lambda^2+1)}{1-\lambda^2} \end{bmatrix} \omega_n = -2h \sum_{j=n}^{\infty} \sin q. \tag{22}$$

Following the same procedure as that used above one can immediately obtain

$$\begin{aligned} q_n &= 2\tan^{-1} \frac{-E_n \pm \{E_n^2 - 4(f_{n+1}\lambda - f_n/\lambda)(f_{n+1}/\lambda - f_n\lambda)\}^{1/2}}{2(f_{n+1} - f_n/\lambda)} \\ E_n &= (1 - U_n U_{n+1})\left(\lambda - \frac{1}{\lambda}\right) \end{aligned} \tag{23}$$

whereas the modified discrete sine–Gordon equation reads as

$$f_{n,t} = 2h\frac{\lambda^2+1}{\lambda^2-1} f_n + (1 + f_n^2)\omega_n. \tag{24}$$

ω_n, given in equation (22), can be easily expressed as a function of f_n, f_{n+1} etc via the Miura map (23). It may be noted that equation (23) was deduced in [6], and we have kept our notation very close to that of [6] for comparison. However, the modified discrete sine–Gordon equation given in (24) was not given there. Also in principle we know the Lax pair of this new equation which are $L_n^{0,1}$ $V_n^{0,1}$.

For the case of the discrete KdV equation

$$q_{n,t} = \exp(-q_{n-1}) - \exp(-q_{n+1}) \tag{25}$$

we note that

$$L_n^{0,0} = \begin{bmatrix} \lambda \exp q_n & (\exp q_{n-1})/\lambda \\ \lambda \exp q_n & \exp q_n/\lambda \end{bmatrix}. \tag{26}$$

So if we assume that $\Delta_n^{+0,0}$ and $L_n^{0,1}$ have the form given in (14) and demand again that $c_n = 0$ we at once obtain

$$q_n = -\log[1 + f_{n+1} - \lambda^2 f_n f_{n+1} - \lambda^2 f_n]. \tag{27}$$

It may be noted that this Miura map is much simpler in form compared to that given in [6].

4. Repeated Miura map

Consider now the case of the Volterra equation written as

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (u_{n+1} - u_{n-1})u_n. \tag{27}$$

In the present case

$$L_n^{0,0} = \begin{bmatrix} \lambda & u_n \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad V_n^{0,0} = \begin{bmatrix} u_n & \lambda u_n \\ -\lambda & -\lambda^2 + u_{n-1} \end{bmatrix}. \tag{28}$$

Set

$$L_n^{0,1} = \begin{bmatrix} b_n & c_n \\ d_n & f_n \end{bmatrix} \qquad \text{and} \qquad \Delta_n^{+0,0} = \begin{bmatrix} 1 & a_n \\ 0 & 1 \end{bmatrix} \tag{29}$$

so that equation (8) leads to

$$\begin{aligned} b_n &= \lambda - a_{n+1} \\ c_n &= u_n - a_n\lambda + a_n a_{n+1} \\ d_n &= -1 \\ f_n &= a_n. \end{aligned} \tag{30}$$

If we set $c_n = 0$, we get

$$u_n = \lambda a_n - a_n a_{n+1} \tag{31}$$

which is the required Miura map. To obtain the modified equation, assume

$$V_n^{0,1} = \begin{bmatrix} v_n & r_n \\ s_n & t_n \end{bmatrix}$$

which, when substituted into equation (10), leads to

$$\begin{aligned} u_n &= u_n - \lambda a_n = -a_n a_{n+1} \\ r_n &= a_{n,t} + \lambda u_n + an(u_{n-1} - \lambda^2) \\ s_n &= -\lambda \\ t_n &= -\lambda^2 + u_{n-1} + \lambda a_n. \end{aligned} \tag{32}$$

Demanding that $r_n = 0$ we get the modified equation

$$a_{n,t} = a_n(\lambda - a_n)(a_{n+1} - a_{n-1}) \tag{33}$$

for which the Lax are can be written as

$$\begin{aligned} L_n^{0,1} &= \begin{bmatrix} \lambda - a_{n+1} & 0 \\ -1 & a_n \end{bmatrix} \\ V_n^{0,1} &= \begin{bmatrix} -a_n a_{n+1} & 0 \\ -\lambda & (\lambda - a_n)(a_{n-1} - \lambda) \end{bmatrix}. \end{aligned} \tag{34}$$

We now demonstrate a new aspect of the present method in which one can start with equation (34), and apply the Gauss decomposition technique to obtain a second Miura map and a new integrable equation.

Assume as before that

$$\Delta_n^{+0,1} = \begin{bmatrix} 1 & \theta_n \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad L_n^{0,2} = \begin{bmatrix} e_n & f_n \\ g_n & h_n \end{bmatrix}$$

and substitute these in (8), so that

$$e_n = \lambda - a_{n+1} - \theta_{n+1}$$
$$f_n = \theta_{n+1}a_n - \lambda\theta_n + \theta_n a_{n+1} + \theta_{n+1}$$
$$g_n = -1$$
$$h_n = a_n + \theta_n.$$

Choosing f_n to be zero yields

$$a_{n+1} = \lambda - \frac{\theta_{n+1}}{\theta_n}a_n - \theta_{n+1} \tag{35}$$

on the other hand equation (10) yields

$$\theta_{n,t} + \theta_n\{(\lambda - a_n)(a_{n-1} - \lambda) + a_n a_{n+1} + \lambda\theta_n\} = 0. \tag{36}$$

One should note that the Miura map (35) nor equations (35) or equation (36) are not explicit in nature, due to the mixing of old and new variables. To remedy this we introduce a new variable

$$\sigma_n = \frac{a_n}{\theta_n} \tag{37}$$

whence (35) gives

$$\theta_{n+1} = \frac{\lambda}{1 + \sigma_n + \sigma_{n+1}} \tag{38}$$

and equation (36) leads to

$$\sigma_{n,t} = \lambda^2 \left\{ \frac{\sigma_n(1+\sigma_n)(\sigma_{n+1} - \sigma_{n-1})}{(1 + \sigma_n + \sigma_{n+1})(1 + \sigma_{n-1} + \sigma_n)} \right\} \tag{39}$$

which is the second Miura-transformed Volterra equation and (38) is the corresponding map.

5. Discussion

Our analysis clearly shows that the Gauss decomposition approach as used in the continuous case can easily be extended to the discrete system and can form an alternative systematic approach. While the present approach can reproduce many of the results obtained by other methodologies, it has the potential to yielding new information about the modified systems. In principle the process of factorization can be repeated many times, though the corresponding results may be algebraically complicated.

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